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Exposita Notes

Theorems on correspondences and stability of the core*

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Summary. In this note two theorems strengthening Grodal's (1971) Theorem on correspondences are proved. The first drops the convexity assumption. The second strengthens that theorem further for the case when the range is the positive orthant. In this case, the conclusion of Grodal's Theorem – the intersection of the integral with the interior of the range being open – is modified to read as the integral being a relative open subset of the positive orthant. An example is provided to show that, such a strengthening is not valid in general. This allows us to dispense with the requirment of convexity of preferences in Grodal's (1971) theorems on the closedness of the set of Pareto optimal allocations, the core, and the continuity of the core correspondence for pure exchange economies. We apply this result to show that blocking coalitions in a large economy are stable.

Keywords and Phrases: Correpondence, Large economy, Core, Pareto set.

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1 Introduction

We start with some notation. Let Ω be the nonnegative orthant in R^l. As usual, $A \triangle A' = (A \setminus A') \cup (A' \setminus A)$ is the symmetric difference of two sets A and A'. For a correspondence $F : T \to \Omega$, where (T, Σ, μ) is a measure space, and a $\mu-$measurable set $A \subset T$ we use short notations $F(A)$ or $\int_A F$ for $\int_A F(t)d\mu(t)$. Instead of $\int_T F$ we write $\int F$. In particular, for a $\mu-$measurable function $f : T \to \Omega$ we write $f(A) = \int_A f(t)d\mu(t)$. The set of all integrable selections of correspondence F we denote as $\mathcal{L}_{\mathcal{F}}$. Relative open subsets of Ω or some other set

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in R^l frequently will be referred to as open sets. Each time it will be clear from the context relative to what set openness is meant. ∂X and $intX$ will denote the boundary and interior of a set X in R^l, respectively. Shorthand 'a. e.' will stand for the phrase 'almost everywhere'.

Grodal (1971, Theorem 2) has proven the following theorem on correspondences.

Theorem 0. *Let (T, Σ, μ) be a measure space and let $X : T \to R^l$ be a measurable convex-valued correspondence. Let furthermore $\varphi : T \to R^l$ be a measurable correspondence such that for a.e. on T, $\varphi(t) \subset X(t)$, $\varphi(t)$ is open relative to $X(t)$ and $\int \varphi d\mu$ is convex. Then*

$$int \left(\int X d\mu \right) \cap \int \varphi d\mu = int \left(\int \varphi d\mu \right).$$

A natural question relating to Theorem 0 is, whether under the assumptions of the theorem, $\int \varphi d\mu$ is relative open in $\int X d\mu$? Clearly, a positive answer would strengthen Grodal's Theorem. Unfortunately, the answer is negative. We bring a due example. However, in the case of $X(t) = \Omega$, a.e. on T, the above question has an affirmative answer (see Theorem 2 below). Furthermore, we show that the assumption of the convexity of $\int \varphi d\mu$ is altogether superfluous (see Theorem 1 below).

Kannai (1970, Theorem B) has proven the continuity of the core of an atomless economy. Grodal (1971, Theorem 4) generalized this result to the case of economies including atoms (mixed economies). The assumptions made by Grodal are somewhat weaker and her proof is based on Theorem 0. She also showed that the set of Pareto optimal allocations is closed in the Banach space of all $\mu-$integrable functions from T into R^l, $L_1(T, \sigma, \mu;\ R^l)$. An analysis of Grodal's proofs shows that the assumption of convexity of preferences is used to ensure the convexity of $\int \varphi$, where φ is an upper contour correspondence for a Pareto optimal or core allocation. This allows an application of Theorem 0. Therefore Theorem 1 makes Grodal's results on closedness of the set of Pareto optimal allocations, and the core, and continuity of the core of pure exchange economies (Grodal, 1971, Theorems 3 and 4) valid even without the convexity requirement on preferences of agents.

Application in a different direction is related to the stability of blocking coalitions. In this note we show that if a coalition blocks an allocation, then any sufficiently small perturbation of that coalition will also block the allocation. In other words, an allocation that is blocked by a coalition will still be blocked by any coalition that slightly differs from the original one. The blocking ability of a coalition is thus robust with respect to small changes in the coalition. Since the unstability of blocking coalitions would reduce the practical relevance of the concept of the core and the precise formation of a particular coalition can hardly be expected in a large economy, such a stability result bears importance. For example, let us consider a large economy, an allocation f, and a coalition S, which is able to block f. Some of the agents who are outside S and are happy with allocation f could destroy coalition S, enticing a tiny group within S by transfer of part of their assignments to them. The robustness of blocking coalitions eliminates this undesirable possibility, hence enhances the practical importance of the concept of core in the context of a large economy. In a sense, our result on the robustness of blocking coalitions, can be thought of as an obverse of Schmeidler's (1972) result, stating that if some coalition blocks an allocation, then there exists a subcoalition with arbitrarily small measure, which also blocks that allocation. Whereas Schmeidler's result emphasizes the decisive power of small coalitions in determining core allocations, we emphasize ineffectiveness of small coalitions in influencing the blocking ability of the coalitions that contain them and are large enough in comparison with them.

We apply our robustness result to strengthen Vind's theorem (1972) on existence of a blocking coalition of an arbitrary measure; we replace the monotonicity of preferences in his theorem by a weaker local insatiability assumption.

We would like to stress that the stability result of this paper and the result on the continuity of the core are of different nature. In particular, for a given economy the latter states *closedness of the core* in $L_1(T, \Sigma, \mu)$, whereas the former states *openness* of blocking coalitions in Σ with respect to the pseudometric $\rho(A,\ B) = \mu(A \triangle B)$ for $A,\ B \in \Sigma$.

2 Theorems on correspondences

For a set A in R^l its convex hull will be denoted *co* A. We will show that the conclusion of Theorem 0 remains valid without the assumption of convexity of $\int \varphi d\mu$.

Theorem 1. *In Theorem 0 the assumption of convexity of integral $\int \varphi d\mu$ can be dropped.*

We preface a proof of this theorem the following simple proposition.

Proposition 1. *Let $X : T \to R^l$ be a measurable convex-valued correspondence such that int $X(t) \neq \emptyset$ a.e. on T. Then*

$$\text{int} \int X = \int \text{int}\, X.$$

Proof. It can be easily shown that $\int \text{int}\, X$ is a nonempty open set. Since $\int \text{int}\, X \subset \int X$ it follows that $\int \text{int}\, X \subset \text{int} \int X$. Show the inverse inclusion. Since $\int \text{int}\, X$ is convex it suffices to show that for an arbitrary point $\bar{x} \in \text{int} \int X$ there is a point from $\int \text{int}\, X$ arbitrarily close to $\bar{x}$. Let $\bar{x} \in \int X$, that is $\bar{x} = \int x$ for some $x \in \mathcal{L}_X$. Fix $y \in \mathcal{L}_{\text{int}\, X}$ and put $x^m(t) = \frac{m-1}{m} x(t) + \frac{1}{m} y(t),\ m \in N$. Clearly $x^m \in \mathcal{L}_{\text{int}\, X}$, $m \in N$, and so $\int x^m \in \int \text{int}\, X$ and $\int x^m \to \int x = \bar{x}$.

Proof of Theorem 1. In the case of a nonatomic measure μ, $\int \varphi$ is convex by Liapunov-Richter Theorem (see Richter, 1963) and then Theorem 0 applies.[1] So, we will assume that T contains an atom. Let $\bar{x}$ belong to int $(\int X) \cap (\int \varphi)$. Then,

[1] In the Appendix we will give a somewhat simpler proof of Theorem 1 for the nonatomic case not appealing to Theorem 0.

there exists $x \in \mathcal{L}_\varphi$ such that $\bar{x} = \int x$. If, $x(t) \in \operatorname{int} \varphi(t)$, or equivalently $x(t) \in \operatorname{int} X(t)$ because of the openness of $\varphi(t)$ in $X(t)$, on a set of positive measure, then by Theorem 1 from (Grodal (1971)), $\bar{x} = \int x$ is an interior point of $\int \varphi$.

Assume now that $x(t) \in \partial X(t)$ a.e. on T, and let T_1 be an atom in T. By Proposition 1 there exists $y \in \mathcal{L}_X$ such that $y(t) \in \operatorname{int} X(t)$ a.e. and $\int y = \bar{x}$. Put $x^m(t) = x(t) + \frac{1}{m}(y(t) - x(t))$ for $t \in T$ and $m \in N$. By Lemma 1 from (Grodal (1971)) $x,\ y,\ \varphi$ and X are constant a.e. on T_1. So $x(t) = x_1,\ y(t) = y_1,\ \varphi(t) = \varphi_1,\ X(t) = X_1$ a.e. on T_1. Let $B_\varepsilon(y_1)$ $(\varepsilon \leq 1)$ be a closed ball contained in X_1. Denote $e(t) = |y(t) - x(t)|^{-1}(y(t) - x(t))$ and put $r(t) = \sup\{r' > 0 :\ [x(t),\ x(t) + r'e(t)] \subset \varphi(t)$ and $r' \leq |x(t) - y(t)|\}$ for $t \in T \setminus T_1$, and $r_1 = \sup\{r' > 0 :\ \operatorname{co}(x_1 \cup B_\varepsilon(y_1)) \cap B_{r'}(x_1) \subset \varphi_1$ and $r' \leq |x_1 - y_1|\}$ for $t \in T_1$. Since $x,\ y$ and φ are measurable, r is measurable and since φ is open-valued and $x(t) \in \varphi(t),\ r(t) > 0$. Moreover, since $r(t) \leq |y(t) - x(t)|$ for $t \in T,\ r(t)$ is integrable.

Obviously, for each $t \in T$, there exists m_t such that $|\frac{1}{m}(y(t) - x(t))| < r(t)$ for $m \geq m_t$. Then $x^m(t) \in [x(t), x(t) + r(t)e(t)] \subset \varphi(t)$ for $m \geq m_t$. Therefore $\delta_m = \mu(E_m)$, where $E_m = \{t \in T :\ |x^m(t) - x(t)| \geq r(t)\}$, converges to zero as $m \to \infty$. By the absolute continuity of integral there exists $\delta > 0$ $(\delta < \mu(T_1))$ such that

$$\int_F |y(t) - x(t)| d\mu(t) < \frac{\varepsilon\mu(T_1)}{2} \quad \text{for each } F \in \Sigma \text{ such that } \mu(F) < \delta. \tag{1}$$

Choose m such that $\delta_m < \delta$, $\frac{\varepsilon}{m} < \frac{r_1}{2}$ and $|x_1^m - x_1| < \frac{r_1}{2}$, where $x_1^m = x^m(t)$ for $t \in T_1$. Clearly, for that m

$$B_{\frac{\varepsilon}{m}}(x_1^m) \subset \varphi(T_1). \tag{2}$$

Put

$$z(t) = \begin{cases} x(t) & \text{for } t \in E_m, \\ x_1^m + \frac{u}{\mu(T_1)} & \text{for } t \in T_1, \\ x^m(t) & \text{otherwise,} \end{cases}$$

where $u = \int_{E_m}(x^m - x)$.

Since $\mu(E_m) < \mu(T_1)$ and T_1 is an atom, $\mu(E_m \cap T_1) = 0$, and so z is correctly defined. Since $|u| \leq \int_{E_m} |x^m - x| = \frac{1}{m}\int_{E_m} |y(t) - x(t)| d\mu(t) < \frac{\varepsilon\mu(T_1)}{2m}$ by (1), by (2) we have $z(t) \in \varphi_1$ for $t \in T_1$. Clearly

$$\int_T z = \int_{E_m} x + \int_{T \setminus (T_1 \cup E_m)} x^m + \int_T x^m = \bar{x}.$$

Thus, we have found a measurable selection z of φ such that $z(t) \in \operatorname{int} \varphi(t)$ for $t \in T_1$. Hence, due to Theorem 1 in (Grodal (1971)), $\bar{x} = \int z$ is an interior point of $\int \varphi$. Theorem 1 is proved.

Theorem 2. *Let (T, Σ, μ) be an arbitrary measure space and $\varphi : T \to \Omega$ be a measurable correspondence with relative open values. Then $\varphi(T)$ is a relative open subset of Ω. In particular, if $x \in \varphi(T)$ and $x > 0$, then $x \in$ int $\varphi(T)$.*

Proof. We consider first the case of atomless measure μ. We start with the "particular case" in the theorem. Suppose $x \in \varphi(T)$ and $x > 0$. Then $x = f(T)$ for some measurable selector $f = (f_1, f_2, \ldots, f_l)$ of φ. Put $T_1 = \{t \in T : f_1(t) > 0\}$. As $f_1(T) = x_1 > 0,\ \mu(T_1) > 0$. Since $\varphi(t)$ is open and φ is measurable, there exists an interval $\{(\alpha, f_2(t), \ldots, f_l(t)) : \ |\alpha - f_1(t)| \le g_1(t)\},$ where $g_1(t) > 0$ is a measurable function defined on T_1, contained in $\varphi(t)$ for $t \in T_1$. Since $f_1(t) = 0$ for $t \in T \setminus T_1,\ (0, f_2(t), \ldots, f_l(t)) = f(t) \in \varphi(t)$ for $t \in T \setminus T_1$. Therefore $\{(\beta, x_2, \ldots, x_l) : \ |\beta - x_1| \le \alpha_1\} \subset \varphi(T)$, where $\alpha_1 = g_1(T_1)$. Similarly, there exist $\alpha_2, \ldots, \alpha_l > 0$, such that a $a_j^{\pm} = (x_1, \ldots, x_j \pm \alpha_j, \ldots, x_l) \in \varphi(T)$ for $j = 2, \ldots, l$. Since by Lyapunov-Richter Theorem $\varphi(T)$ is convex, $C = \mathrm{co}\{a_j^{\pm},\ j = 1, \ldots, l\}$ is contained in $\varphi(T)$. Thus, x has a neighborhood C contained in $\varphi(T)$ i.e., $x \in \mathrm{int}\,\varphi(T)$.

Now we show by induction on the dimension of the space, l, that an arbitrary $x \in \varphi(T)$ is a relative interior point of $\varphi(T)$. Let $l = 1$ and $x \in \varphi(T)$. If $x > 0$, then it is proved above that x is an interior point of $\varphi(T)$. If $x = 0$, then there exists a measurable selector f of φ such that $0 = f(T)$. Since $f(t) \ge 0$ for all t, it follows that $f(t) = 0$ for all t. Hence, $0 \in \varphi(t)$ for all $t \in T$. Therefore, there exists a measurable function $\gamma(t) > 0$ such that $[0, \gamma(t)] \subset \varphi(t)$ for $t \in T$. Consequently, $[0, \varepsilon] \subset \varphi(T)$ for some $\varepsilon > 0$, and then $0 = x$ is a relative interior point of $\varphi(T)$. Suppose the lemma is true for $l \le k - 1$ and prove it for $l = k$. Let $x \in \varphi(T)$. If $x > 0$, then it is proved already that $x \in$ int $\varphi(T)$. Suppose, not $x > 0$. Then some coordinates of x are equal to zero. Without loss of generality, suppose $x_1 = 0$. Let $x = f^0(T)$ for a measurable selector f^0 of φ. Clearly, $f_1^0(t) = 0$ for all $t \in T$. Since $\varphi(t)$ is open in Ω, $\varphi_1(t) = \varphi(t) \cap \Omega_0$, where $\Omega_0 = \{y \in \Omega : y_1 = 0\}$, is open in Ω_0. Since $f^0(t) \in \varphi_1(t)$ for all $t \in T$, it follows from the induction assumption, that $x = f^0(T)$ is a relative interior point of $\varphi_1(T)$. Since $\varphi(t)$ is open in Ω and $f(t) = (0, f_2^\circ(t), \ldots, f_k^\circ(t)) \in \varphi(t)$, then there exists $f_1(t) > 0$ measurable and such that $f(t) = (f_1(t), f_2^\circ(t), \ldots, f_k^\circ(t)) \in \varphi(t)$ for all t. Let $U(x)$ be an arbitrary neighborhood of x in $\varphi_1(T)$ and $\bar{x}_1 = f_1(T)$. Then $U(x) \subset \varphi(T)$ and $\bar{x} = (\bar{x}_1, x_2, \ldots, x_k) \in \varphi(T)$. Since $\varphi(T)$ is convex, it contains $\mathrm{co}(U(x) \cup \{\bar{x}\})$, which is a neighborhood of x in Ω. Thus, the theorem is proved for the case of atomless measure.

We need the following simple claims for the proof of the theorem in the general measure space case.

Claim 1. *If $A,\ B \subset \Omega$ are relative open and $\alpha > 0$, then $A + B$ and αA are relative open.*

Proof of Claim 1 is simple and we omit it.

Denote by N the set of all positive integers. For $A_k \subset \Omega$ $(k \in N)$ and $\alpha_k \in R$ $(k \in N)$ define $\sum_{k \in N} \alpha_k A_k = \{x \in \Omega \ : x = \sum_{k \in N} \alpha_k x^k,\ x^k \in A_k,\ k \in N\}$.

Claim 2. *Let N_0 be an arbitrary subset of N. If $A_k \subset \Omega$ $(k \in N_0)$ are relative open in Ω and $\alpha_k > 0$ $(k \in N_0)$, then $A = \sum_{k \in N_0} \alpha_k A_k$ is relative open in Ω.*

Proof. The case of finite N_0 is a direct consequence of Claim 1. Suppose $N_0 = N$. Let $x \in A$, i. e., $x = \sum_{k \in N} \alpha_k x^k$ for some $x^k \in A_k$ $(k \in N)$. Denote $I = \{1, \ldots, l\}$ and $I_0 = \{i \in I : x_i > 0\}$. Clearly, there exists $n \in N$, such that $\bar{x}_i > 0$ for $i \in I_0$, where $\bar{x} = \sum_{k=1}^{n} \alpha_k x^k$. Since $\bar{x}$ has positive coordinates, it follows that $\alpha_k > 0$ for some $k \in \{1, \ldots, n\}$. Hence, by Claim 1, $\bar{A} = \sum_{k=1}^{n} \alpha_k A_k$ is relative open in Ω. Let $U(\bar{x}) \subset \bar{A}$ be a neighborhood of $\bar{x}$ in Ω. Clearly, $U(\bar{x}) + (x - \bar{x})$ is a neighborhood of x in Ω.

We now proceed with the proof of the theorem. Let $x \in \varphi(T)$ and let f be a measurable selector of φ such that $x = f(T)$. Clearly, $f(T) = f(E_0) + f(E)$, for some $E_0,\ E$, measurable and such that E_0 is contained in the atomless part of T, is $\sigma-$ finite, and $E = \cup_{k \in N_0} E_k$, where E_k $(k \in N_0)$ is a finite or countable set of atoms in T. By Claim 1, it is sufficient to show that $\varphi(E_0)$ and $\varphi(E)$ are relative open in Ω. Relative openness of $\varphi(E_0)$ is proved above, and relative openness of $\varphi(E)$ follows from Claim 2.

Example 1. Let C be the convex hull of the set $C_0 = \{a, b\} \cup C_1$, where $a = (0, 0, 0),\ b = (2, 0, 0)$ and $C_1 = \{x \in R^3 : x_2^2 + (x_3 - 1)^2 = 1,\ x_1 = 1\}$. Let (T, Σ, μ) be the segment $[0, 1]$ with the Lebesgue measure and $\varphi : T \to C$ be defined as

$$\varphi(t) = \begin{cases} C \cap S_1 \text{ if } \ t \in [0,\ 0.5], \\ C \cap S_2 \text{ if } \ t \in (0.5,\ 1], \end{cases}$$

where S_1, S_2 are the open half-spaces $\{x \in R^3 : x_1 < 0.5\}$ and $\{x \in R^3 : x_1 > 1.5\}$, respectively. So φ is a measurable correspondence with relative open in C values. It is easily seen that the point $(1, 0, 0)$ belongs to $\int \varphi$ but it is not a relative interior point of $\int \varphi$ relative to C.

3 Applications to a large economy

1. Theorem 2 implies that in the case of consumption sets $X(t) = \Omega$ Grodal's results on closedness of Pareto set and of the core are valid without the convexity of preferences. We formulate here the corresponding results for a particular case of finite economies.

Theorem 3. *a). Let $E = \{(\omega_i, \succ_i) \in int\Omega \times \mathcal{P}_0,\ i = 1, \ldots, n\}$, where $\mathcal{P}_0$ denotes the set of irreflexive, transitive, continuous and weakly monotone binary relations on Ω. Then the set of Pareto optimal allocations and the core of E, core (E), are closed in R^{ln}.*
b). Let $E_k = \{(\omega_{ik}, \succeq_{ik}) \in int\ \Omega \times \mathcal{P},\ i = 1, \ldots, n\}$ and $E = \{(\omega_i, \succeq_i) \in int\ \Omega \times \mathcal{P},\ i = 1, \ldots, n\}$ be pure exchange economies, where $\mathcal{P}$ is the set of complete continuous preorderings on Ω endowed with the Hausdorff distance. If E_k converges to E, that is $\omega_{ik} \to \omega_i$ and $\succeq_{ik} \to \succeq_i$, $(i = 1, \ldots, n)$ and if $f_k \in core(E_k)$ and $\lim_{k \to \infty} f_k = f$, then $f \in core(E)$.

2. The following results are the applications of Theorem 2 to the stability of blocking coalitions.

Theorem 4. *Let $\mathcal{E} = \{(T, \Sigma, \mu),\ e(t),\ \succ_t,\ t \in T\}$ be a large pure exchange economy with a continuous, locally insatiable and measurable preferences. Then:*
(a) If a coalition A with $e(A) > 0$, blocks an allocation f, then there exists a positive number δ such that every coalition A' with $\mu(A' \triangle A) < \delta$ also blocks f. In particular, in the case $e(t) > 0$ a. e. on T, if a coalition A blocks f, then there exists a positive number δ such that every coalition A' with $\mu(A' \setminus A) < \Delta$ also blocks f,
(b) If a coalition A blocks an allocation f, then there exists a positive number δ such that every coalition $A' \subset A$ with $\mu(A \setminus A') < \delta$ also blocks f.

Corollary. *A feasible allocation in a pure exchange economy $\mathcal{E}$ is the core allocation, if and only if it can not be improved upon by a coalition containing finitely many (in particular, no) atoms.*

Theorem 5. *Let in economy $\mathcal{E}$, (T, Σ, μ) be a finite atomless measure space. Let $e(t) > 0$ a.e. on T be an initial endowment allocation, and $\succ_t$ $(t \in T)$ be continuous, locally insatiable for all $t \in T$, and measurable. Then, if an allocation f is not in the core, then for any number c, such that $0 < c < \mu(T)$, there exists a coalition E, with $\mu(E) = c$, and blocking f.*

4 Proof of Theorems 4 and 5

Proof of Theorem 4. For the sake of simplicity we assume that $\mu(T) < \infty$. Suppose coalition A blocks an allocation f via g^1. Fix $r > 0$ and consider an open ball $B_r(f(t))$ with a center at $f(t)$ and a radius r. Clearly, the correspondence $t \mapsto B_r(f(t))$ is measurable. Put $F(t) = \{x \in \Omega : \ x \succ_t f(t)\}$, and denote by $\partial F(t)$ its relative boundary in Ω. By local insatiability of preferences, $F(t) \cap B_r(f(t))$ is non-empty for each t. Let g^2 be a selector of the restriction of the measurable correspondence $F(\cdot) \cap B_r(f(\cdot))$ into $T \setminus A$. Define g as g^1 on A, and g^2 on $T \setminus A$. Clearly, g is integrable and $g(A) = e(A) \in F(A)$ and $g(t) \succ_t f(t)$ for all t. Denote $\gamma_1(t) = \text{dist}\ (g(t), \partial\ F(t))$ for $t \in T$. Let $\gamma(t)$ be a positive integrable function less than $\gamma_1(t)$. Define $G(t) = \{z \in \Omega : \ ||z - g(t)|| < \gamma(t)\}$ for $t \in T$. G is measurable, nonempty, (relative) open-valued correspondence, such that $\bar{e} = e(A) = g(A) \in G(A)$.

Assume now $\bar{e} = e(A) > 0$, as in the point (a). Then, by Theorem 0, $\bar{e}$ is an interior point of $G(A)$. Let $d > 0$ be such that $B_{3d}(\bar{e}) \subset G(A)$. By the absolute continuity of integral, there exists δ' such that

$$||e(A') - \bar{e}|| < \delta' \ \text{ for } \ \forall A' \in \Sigma \ \text{ such that } \ \rho(A', A) < \delta'. \tag{3}$$

Let $x^i, i = 1, ..., m$, be points in $B_{3d}(\bar{e})$ such that $||x^i - \bar{e}|| = 2d$, $i = 1, ..., m$ and $\text{co}\{x^1, ..., x^m\} \supset B_d(\bar{e})$. Let $h_i \in \mathcal{L}_G$, $i = 1, ..., m$ be such that $h_i(A) = x^i$, $i = 1, ..., m$. Denote by $u_i : T \to \Omega$ a function equal to h_i on A and g^2 on $T \setminus A$ for each $i = 1, ..., m$. Clearly, u_i are selections of correspondence G. By the absolute continuity of integral, there exists $\delta < \delta'$ such that $||u_i(A') - u_i(A)|| < d,\ \forall A'$ such that $\rho(A', A) < \delta,\ i = 1, ..., m$. Therefore

$$co\{u_i(A'),\ i = 1, ..., m\} \supset B_\delta(\bar{e}) \text{ for } A' \text{ such that } \rho(A', A) < \delta. \tag{4}$$

But, since G is a convex-valued correspondence, $G(A')$ is a convex set in Ω. Hence

$$co\{u_i(A'),\ i = 1, ..., m\} \subset G(A'),\ \forall A' \text{ such that } \rho(A', A) < \delta. \tag{5}$$

By (4) and (5), $B_d(\bar{e}) \subset G(A'),\ \forall A'$ such that $\rho(A', A) < \delta$. From that and (3), we have

$$e(A') \in G(A'),\ \forall A' \text{ such that } \rho(A', A) < \delta.$$

So, there exists a function $h : A' \to \Omega$ such that $h(t) \in G(t)$ for all $t \in A'$ and $e(A') = h(A')$. So, A' blocks f via h.

Assume now $\bar{e} \not> 0$. Without loss of generality, assume that only first k coordinates of vector $\bar{e}$ are positive. By Theorem 2, $\bar{e}$ is a relative interior point of $G(A)$ in Ω. Note that for a coalition $A' \subset A,\ e(A') \leq \bar{e} = e(A)$, and therefore $e(A')$ belongs to $\Omega_k = \{y = (y_1, ..., y_l) :\ y_i = 0,\ i = k+1, ..., l\}$. Relative openness of $G(A)$ in Ω implies that $G(A) \cap \Omega_k$ is a relative open set in Ω_k. Applying the above reasoning for the case $\bar{e} > 0$, we will have that there exists $\delta > 0$ such that every coalition A' such that $\rho(A', A) < \delta$ blocks f.

The following example shows that in the case $e(A) \not> 0$, a slightly greater coalition might not be able to block f.

Example 2. Let $T = [0, 3]$ with the Lebesgue measure, and the initial endowment $e : T \to R^2_+$ be defined as

$$e(t) = \begin{cases} (0, 2) \text{ for } t \in [0, 2], \\ (2, 0) \text{ for } t \in (2, 3]. \end{cases}$$

Let preferences be given by the utility functions

$$u_t(x, y) = \begin{cases} y \text{ for } t \in [0, 1], \\ x \text{ for } t \in (1, 3]. \end{cases}$$

and let

$$f(t) = \begin{cases} (0, 1) \quad \text{for } t \in [0, 1], \\ (1, 1.5) \text{ for } t \in (1, 3]. \end{cases}$$

Obviously, $g(t) = e(t)$ for $t \in A = [0, 1]$ blocks f. But coalition $A_\varepsilon = [0, 1 + \varepsilon]$ does not block f for $\varepsilon \in [0, 1]$.

Proof of Corollary. Let a coalition A consisting of infinitely many atoms $A_1, A_2, ...$ and an atomless part A_0 improves upon an allocation f. By Theorem 4 there exists $\delta > 0$ such that any coalition $A' \subset A$ with $\mu(A \setminus A') < \delta$, also blocks f. In particular, any coalition $B = \cup_{k=0}^n A_k$, where integer n satisfies $\sum_{k=n+1}^\infty \mu(A_k) < \delta$, improves upon f.

Proof of Theorem 5. It is a simple application of Lyapunov's Theorem that if a coalition A blocks an allocation f, then for every $c \in (0, \mu(A))$ there exists

a subcoalition $A' \subset A$ of measure c also blocking f (see Schmeidler (1972)). So, it is enough to prove the theorem for c close to $\mu(T)$. Assume, without loss of generality, that $\mu(T) = 1$. Suppose a coalition A blocks f via g. By the assumptions $F(t) = \{x \in \Omega : x \succ_t f(t)\}$ $(t \in T)$ is nonempty and relative open in Ω. By Lyapunov-Richter Theorem (see also Vind (1964, Theorem 2)) the set $F(A)$ is convex. Clearly, it has an interior point and contains vector $e(A) = g(A)$.
Let $\varepsilon \in (0,1)$. By Lyapunov's Theorem, there exists $B \subset T \setminus A$ such that $\mu(B) = (1-\varepsilon)\mu(T \setminus A)$ and $(f-e)(B) = (1-\varepsilon)(f-e)(T \setminus A)$. We will show that $A \cup B$ blocks f. Since $e(A) > 0$ by Theorem 0, $e(A) \in \text{ int } F(A)$. Since $F(A)$ is convex and $e(A) \in \text{ int } F(A)$ and $f(A) \in \partial F(A)$ (by local insatiability of preferences), it follows that $a = \varepsilon e(A) + (1-\varepsilon) f(A) \in \text{ int } F(A)$. Let $\delta > 0$ be such that $B_\delta(a) \subset F(A)$. By local insatiability and measurability of preferences, there exists a measurable function $f^1 : B \to \Omega$, such that $f^1(t) \in F(t)$ for all $t \in B$ and $||f^1(B) - f(B)|| \le \delta$. Since, $b = a - (f^1(B) - f(B)) \in F(A)$, there exists $f^2 : A \to \Omega$, such that $f^2(t) \in F(t)$ for $t \in A$, and $f^2(A) = b$. Define an allocation u as f^1 on B, f^2 on A and e on $T \setminus (A \cup B)$. Then $u(t) \succ_t f(t)$ for all $t \in A \cup B$ and $u(A \cup B) = e(A \cup B)$. Hence $A \cup B$ blocks f via u.

Remark. In proving his theorem Vind (1972) implicitly assumes strict positivity of initial endowments.

Appendix

Proof of Theorem 1 for the nonatomic case. For sake of simplicity we assume that X is a constant correspondence. In this case $\Phi = \int \varphi d\mu$ is convex by Liapunov-Richter Theorem. We will use the following paraphrase of the supporting hyperplane theorem.

Proposition 2. *Let $A \subset R^l$ be convex. Then $\bar{x} \in \text{int} A$ iff for an arbitrary hyperplane H through $\bar{x}$, $A \cap H_i \neq \emptyset$ $(i = 1,2)$ for open half-spaces H_1 and H_2 determined by H.*

Let H be an arbitrary hyperplane through $\bar{x}$ and H_1, H_2 be the two open half-spaces determined by H. By Proposition 2 it is sufficient to show that $\Phi \cap H_i \neq \emptyset$, $i = 1, 2$. Let $T_i = \{t \in T | \, x(t) \in H_i\}, i = 1, 2$. If $\mu(T_i) > 0$ for some $i \in \{1,2\}$, then $\mu(T_i) > 0$ for both $i = 1, 2$. (Otherwise x wouldn't belong to H). As in the proof of Theorem 1 we assume $x(t) \in \partial X$ for all $t \in T$. Since $x(t) \in \partial X$ and $\bar{x} \in \text{int}\, X$ an interval $(x(t), \bar{x})$ is nonempty for each $t \in T$. We put $\varphi_0(t) = (x(t), \bar{x}) \cap \varphi(t)$. Since $\varphi(t)$ is open-valued, $x(t) \in \varphi(t)$ and $(x(t), \bar{x}) \in \text{int}\, X$, $\varphi_0(t)$ is nonempty-valued. Since $\varphi(t)$ and $x(t)$ are measurable, $\varphi_0(t)$ is measurable.

Let $y(t)$ be an arbitrary measurable selection of $\varphi_0(\cdot)$. Put

$$z_i(t) = \begin{cases} y(t) \text{ for } t \in T_i, \\ x(t) \text{ for } t \in T_{i'}, \end{cases}$$

where $i' \neq i$. Then, $\bar{z}_i = \int z_i \in H_i$, $i = 1, 2$.
We now let $\mu(T_i) = 0$, $i = 1, 2$. Then, $x(t) \in H$ for all $t \in T$. Let $B_\delta(x) \subset X$ be

an open ball and let $\varphi_0(t) : T \to X$ be defined as $\varphi_0(t) = \mathrm{co}(B_\delta(x) \cup \{x(t)\}) \cap \varphi(t)$ for $t \in T$. Since, φ is open-valued the following two corresspondences are nonempty-valued: $\varphi_i(t) = \varphi_0(t) \cap H_i,\ i = 1, 2$. Obviously, they are measurable. Let $z_i(t)$ be an arbitrary measurable selector of φ_i. It is clear that $\bar{z}_i = \int z \in H_i,\ i = 1, 2$. So, again there are two points $\bar{z}_i \in H_i,\ i = 1, 2$ of set $\int \phi$ and therefore by Proposition 2, $\bar{x}$ is an interior point of $\int \varphi$.

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